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THE PARADOX OF ANALYSIS

1. INTRODUCTION

Any genuine paradox seems to suggest inevitability of a contradiction. A contradiction itself is not necessarily a paradox: if A implies non-A, we simply infer non-A. A paradox arises if, on the other hand, non-A implies A, since we are bound in this case to infer A: the contradiction is inevitable. A classical example is Russell's paradox of the set of "normal sets". Let a set N be defined as follows:

(1) $\mathrm{N} = \{x;\ x \text{ is not a member of } x\}$

Suppose then that N is a member of N. Because of (1), N is not a member of N: contradiction. But if N is not a member of N, then — again according to (1) — it is a member of N: a new contradiction. It would seem that there is no way out.

There are people whose attitude to paradoxes reduces to a kind of admiration: the world is so complicated and mysterious, they think and even say, that no logical analysis can help us in explaining its paradoxical character. These people become unhappy as soon as a paradox is explained away.

For those of us who can't be content with any *a priori* limitation of rational analysis a paradox is a challenge: all paradoxes can be resolved because they only signalize that there is something wrong with our assumptions. Thus Russell's paradox led Russell to building up his theories of types, which enable us to avoid the "vicious circle" involved in the definition of the set of "normal sets".

The s. c. paradox of analysis is also a challenge. It is "an important and complex problem in the philosophy of logic and language".[1] We will try to show that its resolving is connected with fundamental problems of the logical analysis of natural language and the theory which

[1] G. Bealer: *Quality and Concept*. Clarendon Press Oxford 1982, p. 69.

leads to the most satisfactory solution is Tichý's transparent intensional logic.

2. A FORMULATION OF THE PARADOX OF ANALYSIS

Referring to Bealer[2] we can offer the following general schema of the paradox of analysis (PA):

(2) *X knows that* $A = A$.

(3) *X does not know that* $A = B$.

(4) $A = B$,

where A and B represent some unspecified concepts and (4) (as well as (2)) is a true claim.

One of the instances of PA is

(2′) *Charles knows that 2 = 2.*

(3′) *Charles does not know that 2 = the first prime number.*

(4′) *2 = the first prime number.*

Now we can see that

a) (2) and (3) are compatible,

b) (2) together with (4) implies
that (3) is false,

since (4) justifies substituting B for A in (2).

This is a paradox: after having proved the falsity of (3) (b) we start from this result, ie from "non (3)" and can immediately see that a) does not hold; we know, however, that a) holds, thus (3) must hold, too.

So we ask: what is wrong with our assumptions?

(In this place we let aside Bealer's solution; we will return to it later on.)

3. FREGE'S PROBLEM

There are at least two suspect factors which could be responsible for PA. One of them is a possibly wrong definition of the intuitive notion of meaning, especially as regards the meaning of the particular components of "belief sentences". The second factor (closely related to the first one) could be our wrong analysis of the identity sign in (4). Thus two problems should be solved: the problem I. of the adequate concept of meaning, and the problem II. of the adequate interpretation of sentences containing the identity sign.

Both these problems have been explicitly posed 100 years ago in the famous article by Frege.[3] It was primarily the problem II., which led Frege to his attempt at solving the problem I.[4] He stated the general

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[2] *Ibidem.*

[3] G. Frege: *Über Sinn und Bedeutung. Zeitschrift für philosophische Kritik 100 (1892), 22—5.*

[4] We briefly recapitulate this well-known history, for its recapitulation is necessary for understanding the deep roots of the PA problem and various attempts at its solution.

problem along the following line: comparing sentence 1 of the form $a = a$ with sentence 2 of the form $a = b$, we can see that in the case that sentence 2 is true there is no semantic difference between the two sentences, if the meaning is conceived of as something which obeys the principle of compositionality (as we would put it; this principle — sometimes called "Frege's principle" — has been never explicitly formulated and named by Frege). On the other hand, we feel that the meaning of $a = a$ differs from the meaning of $a = b$: the former sentence — unlike the latter — is analytic; its verification is a logical one, ie trivial. The famous example adduced by Frege is the sentence

(5) *The Morning Star is* (ie =) *the Evening Star.*

Frege's problem can be formulated as follows:

The meaning ("Bedeutung", ie *reference*) of *The Morning Star* as well as of *The Evening Star* is the same: the planet Venus. Thus the sentence (5) seems to be trivially rather than empirically true: it claims the identity of Venus with Venus. The real verification of (5) is, however, empirical — it were astronomers and not logicians who discovered that (5) holds.

Frege's well-known solution to this problem consists in splitting the meaning into two components. Whereas the *reference* ("Bedeutung") concerns the object denoted by the expression (here: Venus), the way in which this object is identified is the *sense* ("Sinn") of this expression. Thus the empirical character of (5) can be explained, since the senses of *The Morning Star* and *The Evening Star* are distinct (these expressions *express* distinct senses).

Now what is the reference of a sentence?

Frege's answer is that a sentence denotes ist truth-value (It was Church[5] who derived this answer via his famous "slingshot argument"), whereas the sense of a sentence is the *thought* ("Gedanke") expressed by this sentence.

Frege's solution can be shown to be unsatisfactory. First, we find nowhere in Frege a rigorous definition of sense. Church's attempt to make this concept logically tractable is ingenious but the criticism of this attempt to be found in Tichý's book[6] shows that what we need is another conceptual apparatus. Second, and this is important especially in the connection with the PA problem, Frege himself was confronted with the possibility that this conception would break down when applied to belief sentences.

Consider the sentence

(6) *Charles believes that the Morning Star is the Evening Star.*

According to the principle of compositionality the meaning of a complex expression E is unambiguously determined by the meanings of the components of E. Thus the reference of (6), ie its truth-value, should remain unchanged if the (true) clause *the Morning Star is the Evening Star*

[5] A. Church: *Introduction to Mathematical Logic.* Princeton 1956.
[6] P. Tichý: *The Foundations of Frege's Logic.* deGruyter 1988.

were substituted for by any true sentence. But this is patently absurd. So Frege had to find a way out unless he wanted to acquiesce in giving up the principle of compositionality.

Frege's way out is remarkable. In such contexts as in (6) (today we would talk about belief sentences or sentences about propositional attitudes or "opaque" or "oblique" or intensional contexts) Frege proposed that the reference of the minor clause were its sense.

This solution is in one respect highly counterintuitive: any sentence would be semantically ambiguous — in some contexts (or isolated) it would denote a truth-value, in other contexts its sense (and this would proceed further dependently on the "degree of embedding"). This is a very unacceptable form of contextualism. On the other hand, an interesting possibility connected with this Frege's idea remained unexploited: why could not an expression (e. g., a sentence) denote its sense in *every* context? This idea would also have suggested that what Frege called "Bedeutung" has nothing to do with the meaning: for example, the truth-value of a sentence can change[7] but the meaning does not change dependently on these changes of truth-values.

4. INTENSIONS

Frege did not exploit Leibniz's idea of possible worlds. His concept of sense, vague as it was, inspired, however, many philosophical logicians, who were ready to explicate this concept in terms of possible worlds. Possible worlds (sets of "possible facts", if you like) are among Montague's *indices,* and a rather great and influential group of "possible-worlds" — semanticists came into being (Kripke, D. Lewis, Hintikka, Kaplan, Tichý etc.).[8] Letting aside details and some (even great) differences between particular members of this group (especially between Tichý's transparent intensional logic (TIL) and the other systems) we can characterize the essential feature of the possible-worlds semantics as follows: "senses" of the linguistic expressions are modelled by *intensions,* viz. functions the domain of which is the *logical space,* ie a set of possible worlds. Thus properties of individuals are distinguished from classes of individuals, since the former are handled as functions which with every possible world (and a time point) associate a class of individuals, „individual concepts" (Tichý: "offices") are conceived of as functions which with every world-time associate at most one individual, etc. The meaning of a sentence is no more its truth-value; it is a proposition, ie a function that with every world-time associates at most one truth-value.

[7] Not for Frege, who would provide any sentence with a "time label" and who called the sentences without this label "incomplete sentences". For the absurd consequences of this view see Tichý's book, p. 187—194.

[8] See, e. g., D. Lewis: *General Semantics.* In: D. Davidson, G. Harman, eds: *Semantics of Natural Language,* Reidel P. C., Dordrecht, Holland, Boston, U. S. A., 1972, p. 169—218; R. Montague: *Pragmatics and Intensional Logic. Ibidem,* p. 142—168; P. Tichý, the cited book.

5. SOLVING PA IN TERMS OF INTENSIONS

Consider the following instance of the PA:

(7) *Charles knows that the Morning Star = the Morning Star.*

(8) *Charles does not know that the Morning Star = the Evening Star.*

(9) *The Morning Star = the Evening Star.*

Clearly, every attempt at resolving the paradox in terms of truth-values (as meanings of sentences) breaks down. We will try to solve it in terms of intensions. Thus *the Morning Star* would denote an individual concept C_1 (see 4.) rather than an individual,[9] *the Evening Star* would denote an idividual concept C_2. Now we can say: a) The identity in (9) is a "contingent identity" in the following sense: C_1 and C_2 are distinct intensions, since x's being the Morning Star does not necessarily imply that y's being the Evening Star means that $x = y$. Thus there are possible worlds where the value of the intension C_1 differs from the value of the intension C_2. But this means that b) the proposition that $C_1 = C_1$, being the constant proposition that returns the value TRUE in each possible world-time differs from the (empirical) proposition that $C_1 = C_2$, since this is no more a constant function.

So the paradox is explained away: knowing (believing, etc.) relates individuals and propositions, so that a substitution *salva veritate* allows only for mutual replacing synonymous sentences, ie sentences denoting (rather than expressing!) one and the same proposition.

6. SOLVING PA IN TERMS OF CONSTRUCTIONS

But, alas, our satisfaction was premature: consider the sentences (2')—(4') or for another example, the following instance of PA:

(10) *Charles knows that equilateral triangles = equilateral triangles.*

(11) *Charles does not know that equilateral triangles = equiangular triangles.*

(12) *Equilateral triangles = equiangular triangles.*

Both these examples can be used as arguments against the solution of PA in terms of intensions. The sentences like (4') and (12) denote (from this "possible-worlds" viewpoint) one and the same proposition, viz. the constant function associating with every world-time the value TRUE. So their semantic status is the same as that of any sentence of the form $a = a$, which is the form of minor clauses in (2') or (10). From the viewpoint of the possible-worlds semantics the respective substitutions are justified and the sentences (3') and (11) are necessarily false, which contradicts the intuitively obvious claim of their compatibility with (2'), (10), respectively. Therefore, we have to find out what is wrong with

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[9] D. Føllesdal: *Situation Semantics and the "slingshot" argument. Erkenntnis* 19 (1983), p. 33, says that there are two kinds of singular terms; the genuine singular terms denote one and the same individual in every possible world, which, of course, is not the case of, e. g., Morning Star or Evening Star.

our assumptions in the case of accepting the "possible-worlds explanation".

Now we can return to Bealer's theory. We refer, e. g., to his already mentioned book, where a well-founded theory has been built up. Since our solution, inspired by TIL, differs from Bealer's, we will not reproduce in full detail Bealer's conception (nor analyze in full detail the reasons for accepting TIL rather than Bealer's PRP — such a critical comparison would require much more place than we have at our disposal); Bealer's main idea is, however, sound and consists in distinguishing between conditions, qualities, connections on the one hand thoughts and concepts on the other hand. The first group corresponds to intensional entities such as propositions, properties, relations-intension;[10] the second group is characterized as containing entities the character of which is inseparably connected with their unique and non-circular definitions. Each of these two groups is then associated with an axiomatic system, so that we have two axiomatic systems, one of them for — *sit venia verbo* — intensions, the other for the entities from the second group. (A unified theory is then described.) The PA is then resolved via taking into account the entities from the second group.

This brief characterization of Bealer's theory is, of course, a simplification. The reader who is interested in details is invited to read the mentioned book (as well as further, more recent articles by the same author), but even now I claim that Bealer's idea of distinguishing between the two groups is important and that my proposal of solution, inspired wholly independently of Bealer's and based on TIL, accepts this distinguishing. Now I can adduce only some global points that justify my not accepting Bealer's solution:

1) I disagree with Bealer as regards his refusing functional idealization of intensions. To say that it is counterintuitive to take a concrete property (e. g., being blue) as a function is unconvincing: every logical idealization (or, if you like, modelling) is just only idealization. Possible--worlds semantics, working with functional idealization, has been able to clear or at least better formulate a great many of problems of philosophical logic.

2) The ontological character of the members of Bealer's second group can be made much more transparent, using direct inductive definitions given in TIL; Bealer's axioms can give us only some logical relations characterizing — *inter alia* — these members. Besides, the semantics of these axioms can be given beforehead (within TIL), which helps us to interpret, e. g., the identity sign so that the axiom $\sim (t_1 = t_2)$ becomes more intelligible. (The construction called *trivialization* in Tichý's book makes this job.)

3) Bealer's proper solution seems to overestimate formal means (brackets).[11]

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[10] Not exactly; Bealer's conception of these entities differs from the possible-worlds conception especially in refusing to model, e. g., properties by functions.

[11] See Tichý's book, p. 134.

Now I go over to a positive solution based on TIL.

Here I would like to stress that what will be informally outlined in the following text can be made perfectly rigorous, using the apparatus defined in Tichý's book. Here I have not got place enough to reproduce this apparatus, based on a (modified Russellian) ramified hierarchy of types and objectually viewed lambda-calculus.

The entities which would probably satisfy the requirements formulated in Bealer's axioms are called *constructions* in TIL. Intuitively, one can say that a construction is a sequence of n steps, $n > 0$, which results or — in strictly defined cases — fails to result in an entity, ie in an object or in a construction. What is important for us just now is that constructions — as extra-linguistic entities — can play the role of meanings of linguistic expressions. A simple example: the expression

(13) $6 : 2$

is analysed in TIL as denoting one of the infinite many ways of constructing the number 3: this "way", ie a construction, consists in applying the dividing operation to the couple <6,2> of numbers. This application of an operation (function) is a *mathematical* construction rather than a linguistic activity: (13) serves only as a linguistic codification of the mathematical activity. The existence of various notational variants (infix notation *vs* prefix notation etc.) proves that what matters is not the notation itself but the abstract (here: mathematical) construction.

We can see that the expression

(14) $\sqrt{9}$

denotes another construction, which, however, constructs the same object as the construction denoted by (13). We say that the respective constructions are *equivalent* (Tichý says "congruent"). Thus equivalent constructions need not be identical.

We are compelled to omit the inductive definition of constructions, as well as of the ramified hierarchy of types; instead we refer to Tichý's book. The above characteristics should be, however, sufficient for informally demonstrating the way the PA can be solved in TIL.

One of the most relevant features of (TIL—) constructions is that they are *extra-linguistic*[12] *structured* entities. Being extra-linguistic makes it possible for us not to be forced into "meta-linguistic manoeuvres" which have been used even by great logicians.[13] Being at the same time structured is a property which in principle makes up the distinction between functions (mappings) — and, therefore, intensions — on the one hand and constructions on the other hand. Properties, individual offices, propositions etc. are — according to the possible-worlds semantics — mappings; as such they lack any information about the way they

[12] The variables are conceived objectually, too, ie as constructions *sui generis*, rather than as letters. Thus the usual symbols x, y, z, ... are not variables: they are names of variables.

[13] Here I am hinting at Bealer's criticism of Church's attempt to resolve the PA in terms of "intensional isomorphism", see A. Church: *Intensional Isomorphism and Identity of Belief. Philosophical Studies V* (1954), p. 65—73.

have been constructed. Thus there are many constructions of one and the same property, proposition etc. (Bealer would say that one and the same property can be given by various distinct concepts, one and the same condition by various distinct thoughts etc.)

Now we can describe the idea of solving the PA in terms of constructions.

The troubles with the unsatisfactory attempts at this solution stem from the theory according to which the meanings of linguistic expressions are typically intensions ("coarse-grained" conceptions). Since the s. c. propositional attitudes (like knowing, believing etc.) surely concern meanings of the minor clauses, we can take them for relations (-in-intensions) between individuals and propositions. We have seen that this conception leads to absurd consequences.[14] What changes, however, if the meaning of an expression is no more an intension, if it is a construction? Then, of course, knowing etc. is a relation (-in-intension) between individuals and constructions (which, for their part, construct propositions or — as in mathematics — truth-values).

Now, let us consider the sentences (2′)—(4′). The construction denoted by

2 = 2

is, of course, not the same construction as that denoted by

2 = the first prime number.

If knowing etc. concerns constructions, then there is no factor which would guarantee that if this relation holds between an individual X and a construction C_1, then it holds also between X and a construction C_2, even if C_2 is equivalent to C_1.

The same argument can be used in any instance[15] of the PA (see, e. g., the sentences (10)—(12)).

This "constructional" view and argument can be formulated in a rigorous way. It is, however, a very natural, very intuitive view — this can be seen, e. g., from the historical fact that for Bolzano[16] the concept of, say, equilateral triangle was another concept than that of equiangular triangle; the Bolzanian concept is an extra-linguistic (and, of course, a non-mental) structured entity.[17]

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[14] The *paradox of omniscience* is one of them: Knowing that a simple mathematicval sentence is true implies (due to this identification of the meaning of a sentence with a proposition) knowing about *any* mathematical sentence whether it is or is not true. Knowing reduces in this case to an attitude to the constant proposition which returns TRUE in every world-time.

[15] With one important exception, see P. Materna: *Three Kinds of Using the Identity Sign.* To appear in *Logica '92*, Prague.

[16] B. Bolzano: *Wissenschaftslehre I.* Sulzbach 1837.

[17] The source of the unjust criticism of Bolzano's conception in Y. Bar-Hillel: *Bolzano's Definition of Analytic Propositions. Methodos II (1950), No 5,* p. 32—55, is the fact that Bar-Hillel did not grasp this deep idea.

7. CONCLUSION

The PA is one of those puzzles which signalize that the s. c. "coarse-grained" conception of meaning is inadequate. To obtain a "fine-grained" conception more proposals have been formulated, among which the most elaborate one is the PRP-theory by Bealer. The author of the present article formulates another conception, which is inspired by Tichý's transparent intensional logic and which is able to attain all the goals which Bealer's theory is intended to attain, without sharing some features of this theory which the author cannot accept. This claim cannot be justified by this brief article alone. Therefore, the author recommends the Reader to compare the PRP-theory with transparent intensional logic, as exposed in Tichý's book *The Foundations of Frege's Logic.*

The author's solution exploits the concept of construction and tries to show (within a limited space) that conceiving meanings as being constructions one satisfies the requirement that meanings should be structured.[18] That the theory of constructions is stronger than the known competing theories (e. g., in that it makes possible quantifying into these structured meanings or that it preserves — unlike Bealer — the principle of extensionality, etc.) has not been possible to show here. Yet it can be shown, when using the apparatus of TIL. In part it has been already made in the cited Tichý's book.

[18] See, e. g., R. Bäuerle, M. J. Cresswell: *Propositional Attitudes.* In: D. Gabbay, F. Guenthner, eds: *Handbook of Philosophical Logic IV.*, p. 491—512.